

Resolution of an integral equation with the Thue-Morse sequence

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Abstract

It is a classical fact that the exponential function is solution of the integral equation $\int_0^X f(x)dx + f(0) = f(X)$. If we slightly modify this equation to $\int_0^X f(x)dx + f(0) = f(\alpha X)$ with $\alpha \in]0, 1[$, it seems that no classical techniques apply to yields solutions. In this article, we consider the parameter $\alpha = 1/2$. We will show the existence of a solution wich takes the values of the Thue-Morse sequence on the odd integers.

1 INTRODUCTION

We consider the functional equation

$$\int_0^X f(x)dx + f(0) = f\left(\frac{X}{2}\right). \quad (1)$$

We can see that the set of continuous solutions is a closed vector space, containing the identically zero function. It is quite clear that any continuous function satisfying Equation (1) is differentiable infinitely many times. So, Equation (1) can be rewritten $f(X) = f'(X/2)/2$.

We can easily verify that the nonzero solutions cannot be expanded in a series. In addition, two solutions equal in a neighborhood of 0 are equal everywhere.

We let τ denote the Thue-Morse substitution. It is a morphism of the free monoid generated by -1 and 1 , defined by $\tau(-1) = (-1)1$ and $\tau(1) = 1(-1)$ and let $\mathbf{u} = (u_n)_{n \geq 0} = (-1)11(-1)1(-1)(-1)1 \dots$ be the Thue-Morse sequence, one of the fixed points of this substitution. See [2, 3, 5] for details.

The aim of this work is to show the following result:

Theorem 1. *There exists a continuous function f_∞ valued in $[-1, 1]$, solution of Equation (1), such that*

- *for each integer n , $f_\infty(2n + 1) = u_n$ and $f_\infty(2n) = 0$;*
- *for each negative real number x , $f_\infty(x) = 0$;*
- *for each positive real number x , $|f_\infty(x)| = |f_\infty(x + 2)|$.*

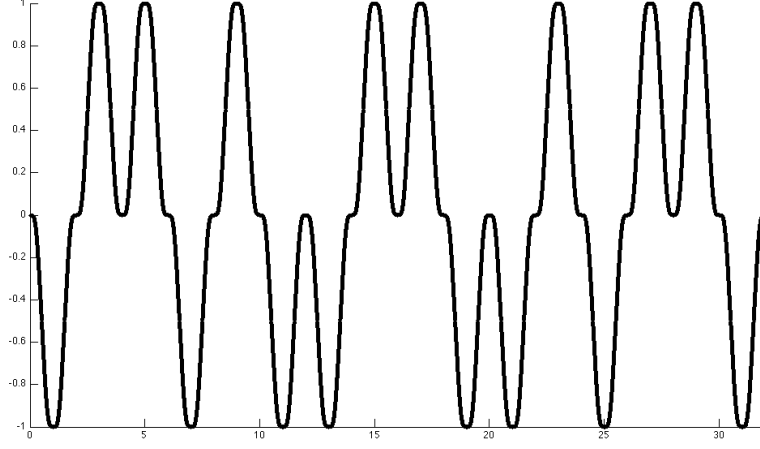


Figure 1: Representation of the graph of f_∞ .

2 INTRODUCTION OF SOME COMBINATORIAL OBJECTS

For any integers $k \geq 0$ and $n \geq 1$, we define the quantities $(\Sigma_n^k)_{(k,n) \in \mathbb{N}^2}$ by

$$\Sigma_0^k = u_k \text{ and } \Sigma_n^0 = 0, \quad (2)$$

and by induction for any integers $k \geq 0$ and $n \geq 0$, by

$$\Sigma_{n+1}^{k+1} = \Sigma_n^k + \Sigma_{n+1}^k. \quad (3)$$

	-1	0	0	0	-----	0	0	-----
	1	-1	0	0	-----	0	0	-----
	1	0	-1	0	-----	0	0	-----
	-1	1	-1	-1	-----	0	0	-----
	1	0	0	-2	-----	0	0	-----
	⋮					⋮	⋮	
$\Sigma_0^k =$	u_k	Σ_1^k	Σ_2^k	-----		Σ_n^k	Σ_{n+1}^k	-----
$\Sigma_0^{k+1} =$	u_{k+1}	Σ_1^{k+1}	Σ_2^{k+1}	-----		Σ_{n+1}^{k+1}		-----

Figure 2: "Pascal's Triangle" associated to the Thue-Morse sequence.

In [7], M. Prunescu has studied the behavior of certain double sequences, called *recurrent two-dimensional sequences* in a more general context. For example when the initialization of the induction given in Equation (2) is

$$\Sigma_0^k = v_k \text{ and } \Sigma_n^0 = w_n,$$

where $(v_n)_n$ and $(w_n)_n$ are sequences such that $v_0 = w_0$. He is particularly interested in the case where $\mathbf{v} = \mathbf{w} = \mathbf{u}$.

If we cleverly renormalize the lines of the standard Pascal triangle, we can approximate a Gaussian curve. We will renormalize the columns of the Pascal triangle associated to the Thue-Morse sequence, to approximate the function f_∞ . We will see that each column is uniformly bounded. This is a very special property of the Thue-Morse sequence.

This property does not hold for Sturmian words, for which the sequence $(\Sigma_2^k)_k$ is not bounded. More precisely, for each parameter $\alpha \in [0, 1]$, we put $\mathbf{v}(\alpha) = (v_n(\alpha))_n$ the sequence defined for each integer n by $v_n(\alpha) = \lfloor (n+1)\alpha \rfloor - \lfloor n\alpha \rfloor$. We associate to the sequence $\mathbf{v}(\alpha)$ the sequence $\mathbf{w}(\alpha) = (w_n(\alpha))_n$ defined for each integer n by $w_n(\alpha) = \alpha$ if $v_n(\alpha) = 0$, and $w_n(\alpha) = -(1-\alpha)$ otherwise. So, the sequence $(\Sigma_1^k)_1$ defined in (3) associated to the sequence $\mathbf{w}(\alpha)$ is bounded. But the sequence $(\Sigma_2^k)_k$ is not bounded. We refer to [1], [4] and [6].

For all integers n , we define a real function f_n , by $f_n(x) = 0$ if $x \leq 0$, and

$$f_n(x) = x_n^k + 2^{n-1}\delta_x(x_n^{k+1} - x_n^k), \text{ if } x = \frac{k}{2^{n-1}} + \delta_x \text{ and } 0 \leq \delta_x < 2^{1-n}$$

for an integer k , with the notation $x_n^k = 2^{-(n-1)(n-2)/2} \cdot \Sigma_n^k$.

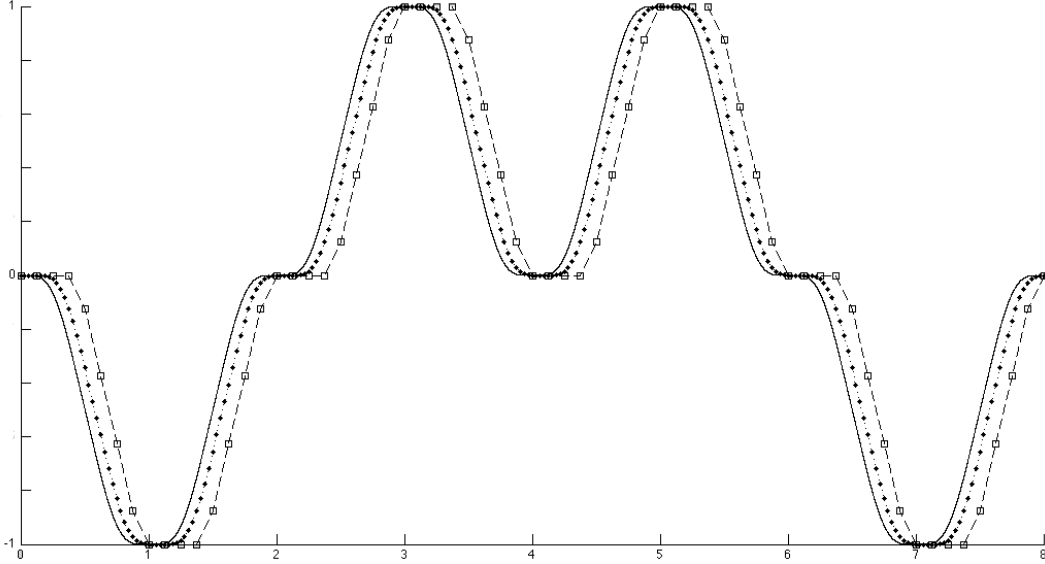


Figure 3: Representation of graph of f_4 , f_6 and f_∞ .

We may also approach this problem from a dynamical point of view. We define T , the application from the set of real sequences into itself by

$$T((y_n)_{n \geq 1}) = \left(y_1, y_2 + y_1, y_3 + \frac{y_2}{2^1}, \dots, y_{n+1} + \frac{y_n}{2^{n-1}}, \dots \right).$$

We must then consider the n -th coordinates of the sequence $(\mathbf{y}^k)_{k \geq 0}$ up to renormalization, where $\mathbf{y}^k = (y_n^k)_{n \geq 1}$, is defined by induction by $\mathbf{y}^0 = \mathbf{0} = (0, \dots, 0, \dots)$, and for each integer $k \geq 1$,

$$\mathbf{y}^{k+1} = T(\mathbf{y}^k) - (u_k, 0, \dots, 0, \dots).$$

3 CALCULATION OF POINTS x_n^k FOR THE FIRST n

We calculate the initial values of sequences $(x_n^k)_k$. To do this, we note that for each integer k , $u_{2k} = u_k = -u_{2k+1}$.

$$\begin{aligned}
 & \left\{ \begin{array}{l} \Sigma_1^{2k} = 0, \\ \Sigma_1^{2k+1} = u_{2k} = u_k, \end{array} \right\} \iff \left\{ \begin{array}{l} x_1^{2k} = 0, \\ x_1^{2k+1} = u_k. \end{array} \right. \\
 & \left\{ \begin{array}{l} \Sigma_2^{4k} = 0, \\ \Sigma_2^{4k+1} = \Sigma_2^{4k} + \Sigma_1^{4k} = 0, \\ \Sigma_2^{4k+2} = \Sigma_2^{4k+1} + \Sigma_1^{4k+1} = u_{2k} = u_k, \\ \Sigma_2^{4k+3} = \Sigma_2^{4k+2} + \Sigma_1^{4k+2} = u_{2k} = u_k, \end{array} \right\} \iff \left\{ \begin{array}{l} x_2^{4k} = 0, \\ x_2^{4k+1} = 0, \\ x_2^{4k+2} = u_k, \\ x_2^{4k+3} = u_k. \end{array} \right. \\
 & \left\{ \begin{array}{l} \Sigma_3^{8k} = 0, \\ \Sigma_3^{8k+1} = \Sigma_3^{8k} + \Sigma_2^{8k} = 0, \\ \Sigma_3^{8k+2} = \Sigma_3^{8k+1} + \Sigma_2^{8k+1} = 0, \\ \Sigma_3^{8k+3} = \Sigma_3^{8k+2} + \Sigma_2^{8k+2} = u_{4k} = u_k, \\ \Sigma_3^{8k+4} = \Sigma_3^{8k+3} + \Sigma_2^{8k+3} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+5} = \Sigma_3^{8k+4} + \Sigma_2^{8k+4} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+6} = \Sigma_3^{8k+5} + \Sigma_2^{8k+5} = 2u_{4k} = 2u_k, \\ \Sigma_3^{8k+7} = \Sigma_3^{8k+6} + \Sigma_2^{8k+6} = u_k, \end{array} \right\} \iff \left\{ \begin{array}{l} x_3^{8k} = 0, \\ x_3^{8k+1} = 0, \\ x_3^{8k+2} = 0, \\ x_3^{8k+3} = u_k/2, \\ x_3^{8k+4} = u_k, \\ x_3^{8k+5} = u_k, \\ x_3^{8k+6} = u_k, \\ x_3^{8k+7} = u_k/2. \end{array} \right. \\
 & \left\{ \begin{array}{l} \Sigma_4^{16k} = 0, \\ \Sigma_4^{16k+1} = \Sigma_4^{16k} + \Sigma_3^{16k} = 0, \\ \Sigma_4^{16k+2} = \Sigma_4^{16k+1} + \Sigma_3^{16k+1} = 0, \\ \Sigma_4^{16k+3} = \Sigma_4^{16k+2} + \Sigma_3^{16k+2} = 0, \\ \Sigma_4^{16k+4} = \Sigma_4^{16k+3} + \Sigma_3^{16k+3} = u_k, \\ \Sigma_4^{16k+5} = \Sigma_4^{16k+4} + \Sigma_3^{16k+4} = 3u_k, \\ \Sigma_4^{16k+6} = \Sigma_4^{16k+5} + \Sigma_3^{16k+5} = 5u_k, \\ \Sigma_4^{16k+7} = \Sigma_4^{16k+6} + \Sigma_3^{16k+6} = 7u_k, \\ \Sigma_4^{16k+8} = \Sigma_4^{16k+7} + \Sigma_3^{16k+7} = 8u_k, \\ \Sigma_4^{16k+9} = \Sigma_4^{16k+8} + \Sigma_3^{16k+8} = 8u_k, \\ \Sigma_4^{16k+10} = \Sigma_4^{16k+9} + \Sigma_3^{16k+9} = 8u_k, \\ \Sigma_4^{16k+11} = \Sigma_4^{16k+10} + \Sigma_3^{16k+10} = 8u_k, \\ \Sigma_4^{16k+12} = \Sigma_4^{16k+11} + \Sigma_3^{16k+11} = 7u_k, \\ \Sigma_4^{16k+13} = \Sigma_4^{16k+12} + \Sigma_3^{16k+12} = 5u_k, \\ \Sigma_4^{16k+14} = \Sigma_4^{16k+13} + \Sigma_3^{16k+13} = 3u_k, \\ \Sigma_4^{16k+15} = \Sigma_4^{16k+14} + \Sigma_3^{16k+14} = u_k, \end{array} \right\} \iff \left\{ \begin{array}{l} x_4^{16k} = 0, \\ x_4^{16k+1} = 0, \\ x_4^{16k+2} = 0, \\ x_4^{16k+3} = 0, \\ x_4^{16k+4} = u_k/8, \\ x_4^{16k+5} = 3u_k/8, \\ x_4^{16k+6} = 5u_k/8, \\ x_4^{16k+7} = 7u_k/8, \\ x_4^{16k+8} = u_k, \\ x_4^{16k+9} = u_k, \\ x_4^{16k+10} = u_k, \\ x_4^{16k+11} = u_k, \\ x_4^{16k+12} = 7u_k/8, \\ x_4^{16k+13} = 5u_k/8, \\ x_4^{16k+14} = 3u_k/8, \\ x_4^{16k+15} = u_k/8. \end{array} \right.
 \end{aligned}$$

4 FIRST COMBINATORIAL RESULTS

Lemma 1. For any integers $n \geq 1$, $k \geq 0$ and $l \in \{0, \dots, 2^n - 1\}$, there exists $a(n, l)$, which does not depend on k , such that $\Sigma_n^{2^n k + l} = a(n, l)u_k$. In particular, $\Sigma_n^{2^n k} = a(n, 0) = 0$. For any integer $n \geq 1$ and $l \in \{0, \dots, 2^n - 1\}$, the coefficients $a(n, l)$ satisfy the following relation:

$$\begin{aligned}
 a(n+1, l+1) &= a(n+1, l) + a(n, l) \\
 \text{and } a(n+1, l+2^n+1) &= a(n+1, l+2^n) - a(n, l).
 \end{aligned} \tag{4}$$

We conclude that $a(n+1, l+2^n) = a(n+1, l) - a(n, l)$.

Proof. We have seen in Section 3, that this result is true for the first values of the integer n . We suppose that the result is true up to a rank $n - 1$ and we will show that it is still true up to order n . We start by verifying that $\Sigma_{n+1}^{2^n k}$ is zero for each integer k :

$$\begin{aligned}
\Sigma_n^{2^n k} &= \sum_{l=0}^{2^n k-1} \Sigma_{n-1}^l + \Sigma_{n+1}^0 = \sum_{j=0}^{k-1} \sum_{l=0}^{2^n-1} \Sigma_{n-1}^{2^n j+l}, \\
&= \sum_{j=0}^{k-1} \left(\sum_{l=0}^{2^{n-1}-1} \Sigma_{n-1}^{2^{n-1}(2j)+l} + \sum_{l=0}^{2^{n-1}-1} \Sigma_{n-1}^{2^{n-1}(2j+1)+l} \right), \\
&= \sum_{j=0}^{k-1} \left(u_{2j} \sum_{l=0}^{2^{n-1}-1} a(n-1, l) + u_{2j+1} \sum_{l=0}^{2^{n-1}-1} a(n-1, l) \right), \\
&= \left(\sum_{l=0}^{2^{n-1}-1} a(n-1, l) \right) \cdot \left(\sum_{j=0}^{k-1} u_{2j} + u_{2j+1} \right), \\
&= 0.
\end{aligned}$$

Now, we focus on the recurrence relations verified by the coefficients $a(n, k)$. The integer n is already fixed, we show this result by induction on l and k . For $l = 0$, we have seen that this result was true for all integers k . Suppose Equation (4) holds for all k up to a rank l and show that it is still true for all k the rank $l + 1$.

$$\begin{aligned}
\Sigma_n^{2^n k+l+1} &= \Sigma_n^{2^n k+l} + \Sigma_{n-1}^{2^{n-1}(2k)+l} = a(n, l)u_k + a(n-1, l)u_{2k}, \\
&= a(n, l)u_k + a(n-1, l)u_k = \left(a(n, l) + a(n-1, l) \right) u_k. \\
\\
\Sigma_n^{2^n k+2^{n-1}+l+1} &= \Sigma_n^{2^n k+2^{n-1}+l} + \Sigma_{n-1}^{2^{n-1}(2k+1)+l}, \\
&= a(n, l+2^{n-1})u_k + a(n-1, l)u_{2k+1}, \\
&= a(n, l)u_k - a(n-1, l)u_k, \\
&= \left(a(n+2^{n-1}, l) - a(n-1, l) \right) u_k.
\end{aligned}$$

Then, we verify the last relation of the lemma:

$$\begin{aligned}
a(n+1, l+2^n) &= a(n+1, l+2^n-1) - a(n, l-1), \\
&= a(n+1, l+2^n-2) - a(n, l-2) - a(n, l-1), \\
&= a(n+1, 2^n) - \sum_{j=0}^{l-1} a(n, j).
\end{aligned}$$

We get $a(n+1, l+2^n) = a(n+1, 2^n) - a(n+1, l)$. □

Lemma 2. For any integer n , $a(n, 2^{n-1}) = 2^{(n-1)(n-2)/2}$.

Proof. Since $a(1, 1) = 1$, this result is immediate by induction from the relation:

$$\begin{aligned}
a(n+1, 2^n) &= \sum_{l=0}^{2^n-1} a(n, l) = \sum_{l=0}^{2^{n-1}-1} a(n, l) + \sum_{l=0}^{2^{n-1}-1} a(n, l+2^{n-1}), \\
&= \sum_{l=0}^{2^{n-1}-1} a(n, l) + \sum_{l=0}^{2^{n-1}-1} \left(a(n, 2^{n-1}) - a(n, l) \right), \\
&= \sum_{l=0}^{2^{n-1}-1} a(n, 2^{n-1}) = 2^{n-1} a(n, 2^{n-1}).
\end{aligned}$$

So, $a(n+1, 2^n) = 2^{n-1} \cdot 2^{(n-1)(n-2)/2} = 2^{(n+1-1)(n+1-2)/2}$. □

Lemma 3. For every integer n , and $l \in \{0, \dots, 2^n - 1\}$,

$$0 \leq a(n, l) \leq 2^{(n-1)(n-2)/2}. \quad (5)$$

Proof. We will show this by induction on the integer n . We initialized the recurrence. We suppose that the result is true up to the rank n and show that it is still true to the rank $n + 1$.

Suppose then that for each integer $l \in \{0, \dots, 2^n - 1\}$, Equation (5) holds. Since for every $l \in \{0, \dots, 2^n - 1\}$,

$$a(n+1, l+1) = a(n, l) + a(n+1, l) \geq 0,$$

the sequence $\left(a(n+1, l)\right)_{l \in \{0, \dots, 2^n\}}$ increases from 0 to $2^{(n-1)(n-2)/2}$ for $l = 2^n$. We can then conclude because if $l \in \{0, \dots, 2^n - 1\}$,

$$0 \leq a(n+1, l+2^n) = 2^{(n-1)(n-2)/2} - a(n+1, l) \leq 2^{(n-1)(n-2)/2}. \quad \square$$

Lemma 4. For every integer n , and $l \in \{0, \dots, 2^{n-2} - 1\}$,

$$a(n, 2l+1) \geq a(n, 2l) \geq 2^{n-2}a(n, l).$$

Proof. We prove this lemma by induction on n . For $n = 1$, the result is immediate. We show that if the result is true up to the rank n , it is still true to the rank $n + 1$. We show this by induction on l . From Lemma 1, this is true for $l = 0$ and $l = 1$. We suppose that the result is true for $2l$ and $2l + 1$, and we show that it is still true for $2l + 2$ and $2l + 3$.

If $l \in \{0, \dots, 2^{n-2} - 1\}$, then $a(n, 2l) \leq a(n, 2l + 1)$ and

$$\begin{aligned} a(n+1, 2(l+1)+1) &\geq a(n+1, 2(l+1)) \\ &\geq a(n+1, 2l) + a(n, 2l) + a(n, 2l+1) \\ &\geq 2^{n-1}a(n, l) + a(n, 2l) + a(n, 2l+1) \\ &\geq 2^{n-1}a(n, l) + 2a(n, 2l) \\ &\geq 2^{n-1}a(n, l) + 22^{n-2}a(n-1, l) \\ &\geq 2^{n-1}\left(a(n, l) + a(n-1, l)\right) \\ &\geq 2^{n-1}a(n, l+1). \end{aligned}$$

If $l \in \{2^{n-2}, \dots, 2^{n-1} - 1\}$, then $a(n, 2l) \geq a(n, 2l + 1)$ and

$$\begin{aligned} a(n+1, 2(l+1)+1) &\geq a(n+1, 2(l+1)) \\ &\geq a(n+1, 2l) + a(n, 2l) + a(n, 2l+1) \\ &\geq 2^{n-1}a(n, l) + a(n, 2l) + a(n, 2l+1) \\ &\geq 2^{n-1}a(n, l) + 2a(n, 2l+1) \\ &\geq 2^{n-1}a(n, l) + 22^{n-2}a(n-1, l) \\ &\geq 2^{n-1}\left(a(n, l) + a(n-1, l)\right) \\ &\geq 2^{n-1}a(n, l+1). \end{aligned}$$

\square

5 PROOF OF THEOREM 1

Let us start by proving the following lemma.

Lemma 5. *Let n be an integer greater than or equal to 1.*

1. *For each integer m , $f_n(2m+1) = u_m$ and $f_n(2m) = 0$.*
2. *For each real x , $f_n(x) \in [-1, 1]$.*
3. *For each integer m and for each $x \in [0, 2]$,*

$$f_n(x+2m) = -f_n(x)u_m. \quad (6)$$

4. *For each integer m , if $u_m = -1$, f_n increases on $[m, m+1]$, and if $u_m = 1$, f_n decreases on $[m, m+1]$. In particular, f_n and u_m have the same sign on $[2m, 2m+2]$.*
5. *For each couple of reals $(x, y) \in [0, 2]^2$: $|f_n(x) - f_n(y)| \leq |x - y|$.*
6. *For each real $x \in [0, 1]$, the sequence $(f_n(x))_n$ decreases.*
7. *For each real $x \in [2m, 2m+1]$, $(f_n(x))_n$ is decreasing if $u_m = -1$, and increasing otherwise. And for each real $x \in [2m+1, 2m+2]$, $(f_n(x))_n$ is increasing if $u_m = -1$, and decreasing otherwise*

Proof of Point 1 of Lemma 5. We fix an integer $n \geq 1$, and an integer m . By the definition of functions f_n , $f_n(2m) = \sum_n^{2^n m} 2^{-(n-1)(n-2)/2} = 0$ and

$$f_n(2m) = \sum_n^{2^n m + 2^{n-1}} 2^{-(n-1)(n-2)/2} = 2^{(n-1)(n-2)/2} u_m 2^{-(n-1)(n-2)/2} = u_m. \quad \square$$

Proof of Point 2 of Lemma 5. From Lemma 3, for each positive real x :

$$\begin{aligned} |f_n(x)| &\leq \sup \{ |x_n^{2^n k + l}|; k \in \mathbb{N} \text{ and } l \in \{0, \dots, 2^n\} \} \\ &\leq \sup \{ |\sum_n^{2^n k + l} 2^{-(n-1)(n-2)/2}|; k \in \mathbb{N} \text{ and } l \in \{0, \dots, 2^n\} \} \\ &\leq \sup \{ a(n, l) 2^{-(n-1)(n-2)/2}; l \in \{0, \dots, 2^n\} \}. \end{aligned}$$

From Lemma 3, $|a(n, l) 2^{-(n-1)(n-2)/2}| \leq 1$ and $|f_n(x)| \leq 1$. \square

Proof of Point 3 of Lemma 5. We fix a real $x = \frac{k}{2^{n-1}} + \delta_x \in [0, 2]$, and an integer m .

$$\begin{aligned} f_n(x+2m) &= f_n\left(\frac{k+m2^n}{2^{n-1}} + \delta_x\right) = x_n^{k+m2^n} + \delta_x(x_n^{k+m2^n+1} - x_n^{k+m2^n}) \\ &= -u_m \left(x_n^{k+m2^n} + \delta_x(x_n^{k+m2^n+1} - x_n^{k+m2^n}) \right) = -u_m f_n(x). \end{aligned}$$

We treat now the case where $x = 2m$. From Point 1, $f_n(2m) = f_n(0) = 0$ and this point is demonstrated. \square

Proof of Point 4 of Lemma 5. We verify this result by induction on n . For $n = 1$, the result is true. Now, we verify that if it is true up to the rank $n-1$, it will be still true to the rank n . For any integer k ,

$$\begin{aligned} f_n\left(\frac{k}{2^{n-1}}\right) - f_n\left(\frac{k+1}{2^{n-1}}\right) &= x_n^k - x_n^{k+1} = (\sum_n^k - \sum_n^{k+1}) 2^{-(n-1)(n-2)/2}, \\ &= \sum_{n-1}^k 2^{-(n-1)(n-2)/2} = \sum_{n-1}^k 2^{-(n-1-1)(n-1-2)/2} 2^{-n+1}, \\ &= x_{n-1}^k 2^{-n+1} = f_{n-1}\left(\frac{k}{2^{n-2}}\right) 2^{-n+1}. \end{aligned}$$

So, if $k = m2^{n-1} + l$, with $0 \leq l \leq 2^{n-1} - 1$, then $\frac{k}{2^{n-2}} = \frac{l}{2^{n-2}} + 2m$, and $f_{n-1}(\frac{k}{2^{n-2}})$ and u_m have the same sign. Then, f_n is decreasing on $[m, m+1]$ if $u_m = -1$, and increasing otherwise.

And if $k = (m+1)2^{n-1} + l$, with $0 \leq l \leq 2^{n-1} - 1$, then $\frac{k}{2^{n-2}} = \frac{l}{2^{n-2}} + 2(m+1)$, and $f_{n-1}(\frac{k}{2^{n-2}})$ have the same sign as u_{m+1} , and so f_n is decreasing on $[m, m+1]$ if $u_{m+1} = -1$, and increasing otherwise.

We suppose now that $u_m = u_{2m} = -1$. The function f_n decreases from 0 to -1 on $[2m, 2m+1]$, and increases from -1 to 0 on $[2m+1, 2m+2]$. So, the function is negative on $[2m, 2m+2]$. We can then use the same argument if $u_m = 1$ to complete the proof of this point. \square

Proof of Point 5 of Lemma 5. We fix two reals x and y , such that $x \leq y$, verifying:

$$x = \frac{k}{2^{n-1}} + \delta_x \text{ and } y = \frac{l}{2^{n-1}} + \delta_y,$$

where δ_x and δ_y are less than $1/2^{n-1}$.

$$\begin{aligned} f_n(x) - f_n(y) &= x_n^k + 2^{n-1}\delta_x(x_n^{k+1} - x_n^k) - x_n^l - 2^{n-1}\delta_y(x_n^{l+1} - x_n^l), \\ &= x_n^k - x_n^{k+1} + \dots + x_n^{l-1} + x_n^l + 2^{n-1}\delta_x(x_n^{k+1} - x_n^k) - 2^{n-1}\delta_y(x_n^{l+1} - x_n^l), \\ &= (f_{n-1}(\frac{k}{2^{n-2}}) + \dots + f_{n-1}(\frac{l-1}{2^{n-1}}))2^{1-n} - \delta_x f_{n-1}(\frac{k}{2^{n-2}}) + \delta_y f_{n-1}(\frac{l}{2^{n-2}}). \end{aligned}$$

Since f_{n-1} is negative on $[0, 1]$:

$$|f_n(x) - f_n(y)| \leq \frac{l-k}{2^{n-1}} + \delta_y - \delta_x = |x - y|. \quad \square$$

Proof of Point 6 of Lemma 5. We show that for each integer N and each integer $l \in \{0, \dots, 2^{n-1} - 1\}$, the sequence $(f_{n+N+1}(\frac{l}{2^{N-1}}))_n$ is decreasing:

$$\begin{aligned} f_{n+N+1}(l/2^{N-1}) &= f_{n+N+1}(\frac{l2^n}{2^{N+n-1}}) = f_{n+N+1}(\frac{l2^n-1}{2^{N+n-1}}) = x_{N+n+1}^{l2^n}, \\ &= \sum_{N+n+1}^{l2^n} 2^{(N+n)(N+n-2)} = a(N+n+1, l2^n)2^{(N+n)(N+n-2)}u_0, \\ &= -a(N+n+1, l2^n)2^{(N+n)(N+n-2)}, \\ &\leq -2^{N+n-1}a(N+n, l2^{n-1})2^{(N+n)(N+n-1)/2}. \end{aligned}$$

This result is then proved because

$$\begin{aligned} -2^{N+n-1}a(N+n, l2^{n-1})2^{(N+n)(N+n-1)/2} &= -a(N+n, l2^{n-1})2^{(N+n-1)(N+n-2)/2}, \\ &= a(N+n, l2^{n-1})2^{(N+n-1)(N+n-2)/2}u_0, \\ &= f_{n+N}(l/2^{N-1}). \end{aligned}$$

Then, $f_{n+N+1}(l/2^{N-1}) \leq f_{n+N}(l/2^{N-1})$. \square

Proof of Point 7 of Lemma 5. Let $x \in [0, 1]$ and m be an integer. We deduce the proof of this point from the following remarks:

$$f_n(x+1) = -1 - f_n(x) \text{ and } f_n(x+2m) = -u_m \cdot f_n(x). \quad \square$$

Proof of Theorem 1. For each real number x , the sequence $(f_n(x))_n$ is monotone and bounded, so it converges. Let $f_\infty(x)$ denote the limit. It is clear that the function f_∞ is 1-Lipschitz. The third point is proved from Equation (6). With the previous lemma, we can deduce that the range of the function f_∞ is included in $[-1, 1]$ and that for each positive integer m :

$$f_\infty(2m) = 0 \text{ and } f_\infty(2m+1) = u_m.$$

We need to verify that it is a solution of Equation (1). We fix a positive real $X \in [2m, 2m+2[$ and an integer n such that $[X, X+1/2^{n-1}] \subset]2m, 2m+2[$. We fix l , the integer such that $0 \leq \delta_X = X - l/2^{n-1} \leq 1/2^{n-1}$.

Then for any integer m sufficiently large:

$$\begin{aligned} f_{n+m+1}(X) &= f_{n+m+1}\left(\frac{l2^m}{2^{n+m-1}}\right) + f_{n+m+1}(X) - f_{n+m+1}\left(\frac{l}{2^{n-1}}\right), \\ &= 2^{1-n-m} \sum_{j=0}^{l2^m-1} f_{n+m}\left(\frac{j}{2^{n+m}}\right) + f_{n+m+1}(X) - f_{n+m+1}\left(\frac{l}{2^{n-1}}\right). \end{aligned}$$

Since for each real x , the sequence $(f_n(x))_n$ is monotone, we let m tend to infinity to find:

$$f_\infty(X) = \int_0^{2\frac{l}{2^{n-1}}} f_\infty(x)dx + f_\infty(X) - f_\infty\left(\frac{l}{2^{n-1}}\right).$$

We deduce therefore that

$$\left| f_\infty(X) - \int_0^{2X} f_\infty(x)dx \right| \leq 2 \left| f_\infty(X) - f_\infty\left(\frac{l}{2^{n-1}}\right) \right| \leq \frac{1}{2^{n-2}}.$$

Then, when n goes to infinity, $f_\infty(X) = \int_0^{2X} f_\infty(x)dx$. □

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